Pergamon

Int. J. Solids Structures Vol. 35, No. 22, pp. 2877–2899, 1998

0020-7683/98 $19.00 + .00

PII: S0020-7683(97)00356-9

VIBRATIONS OF COMPOSITE CIRCULAR CYLINDRICAL VESSELS

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(*Received* 17 *April* 1997; *in revised form* 18 *November* 1997)

Abstract—Free vibrations are analyzed for a laminated composite circular cylindrical vessel consisting of a laminated composite circular cylindrical shell and laminated composite circular plate lids with a hole at both ends. In the analysis, by using the exact solutions of the equations of motion for the circular cylindrical shell and the circular plate, the Lagrangian of the vessel is expressed in a quadratic form of the unknown boundary values. The frequency equations are obtained by minimizing the Lagrangian with respect to the unknown boundary values.

Numerical studies are made for vessels with symmetric cross-ply laminates of unidirectional graphite fiber reinforced epoxy. The natural frequencies, the mode shapes and the bending moment distributions are obtained and their characteristics are investigated. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Investigations on vibrations of homogeneous and isotropic combined shell structures have been started in earnest from the early years of the 1960s. As the main dynamic models (or analytic models) were taken to be combined systems of circular plate–circular cylinder, circular cylinder–conical shell, circular cylinder–circular cylinder, circular cylinder–spherical shell and so forth. The vibration analysis of combined shell structures is very complicated and troublesome, and so many solution techniques such as the Lagrangian minimizing method, the finite element method, the substructure synthesis method and the transfer matrix method have been devised and applied (good summaries on these subjects have been given in Hirano, 1969; Tavakoli and Singh, 1989; and Huang and Soedel, 1993). The theoretical analyses for vibrations of laminated composite shells have been practised very often lately but, at present, they are not for practical combined shells or vessels but are mostly for simple shells.

In view of the circumstances, in this paper, free vibrations are analyzed for a laminated composite circular cylindrical vessel consisting of a laminated composite circular cylindrical shell and laminated composite circular plate lids with a hole at both ends. In the process of analysis, at first, the equations of motion for the circular cylindrical shell and the circular plate are solved exactly by using power series expansions. Then by using the solutions obtained, each Lagrangian of vibration, of the circular cylindrical shell and the corresponding circular plates, is expressed in quadratic forms of the boundary values of the displacements and the slopes. Next, by using the geometrical conditions of continuity at connecting points, relations are obtained among the boundary values of the circular cylindrical shell and the circular plates. Moreover, by considering the boundary conditions, the Lagrangian of the combined system is expressed in a quadratic form of the unknown boundary values. The frequency equations are obtained by minimizing the Lagrangian with respect to the unknown boundary values.

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Numerical studies are made for circular cylindrical vessels with symmetric cross-ply laminates of unidirectional graphite fiber reinforced epoxy. The natural frequencies, the mode shapes and the bending moment distributions are obtained and their characteristics are investigated.

2. ANALYSIS

2.1. *Lagrangian of a laminated cross-ply circular cylindrical shell in terms of boundary values*

Figure 1 shows the middle surface of a thin circular cylindrical shell and the co-ordinate system, with x' being the axial, θ the circumferential and z the radial co-ordinate parameters. The origin is taken to be at the center O of the middle cross-section. The length, mean radius and thickness are denoted by $2l$ $(= 2\,\mu R_0)$, R_0 and h, respectively.

Figure 2 shows the cross-sectional view of the shell, in which h_k and h_{k-1} are the values of the normal co-ordinate, measured from the middle surface, at the outer and inner surfaces of the kth laminate, respectively.

Employ a nondimensional coordinate $x = x'/R_0$ and denote the displacements in the x', θ and z directions by $[u(x,\theta),\ v(x,\theta),\ w(x,\theta)]\sin pt$, where p and t are the circular frequency and the time. The Lagrangian of a laminated cross-ply thin circular cylindrical

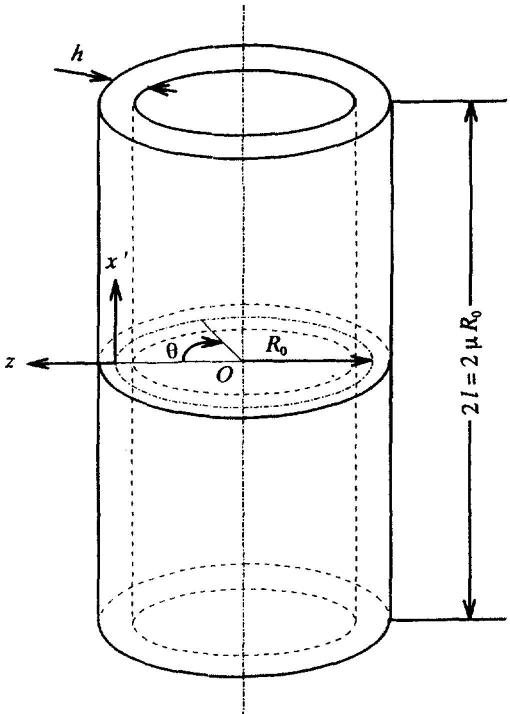

Fig. 1. Middle surface of a circular cylindrical shell and co-ordinate system.

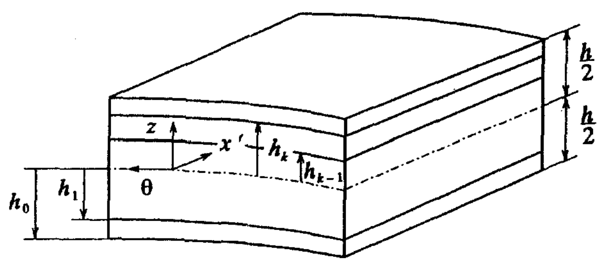

Fig. 2. Cross-sectional view of shell.

shell composed of N laminae symmetric about the middle surface in a period of vibration is given from the authors' previous paper (Hu *et al.*, 1996) as follows:

$$-L\Big/\frac{D}{2R_0^2} = \iint \Bigg\{ \alpha^4 H_1 (u^2+v^2+w^2)$$
$$+\beta^2 \left[A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + 2A_{12}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial \theta}+w\right) + A_{22}\left(\frac{\partial v}{\partial \theta}+w\right)^2 + A_{66}\left(\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}\right)^2 \right]$$
$$+D_{11}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial \theta}-v\right)$$
$$+D_{22}\left(\frac{\partial^2 w}{\partial \theta^2}-w\right)^2 + D_{66}\left[3\left(\frac{\partial^2 w}{\partial x\,\partial \theta}-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial^2 w}{\partial x\,\partial \theta}+\frac{\partial u}{\partial \theta}\right)^2\right]\Bigg\}\,\mathrm{d}x\,\mathrm{d}\theta \tag{1}$$

where

$$D = E_0 h^3, \quad \beta = R_0/h, \quad \alpha^4 = \rho_0 h p^2 R_0^4/D$$

$$H_1 = \sum_{k=1}^{N}\frac{\rho^{(k)}(h_k-h_{k-1})}{\rho_0 h}, \quad A_{ij} = \sum_{k=1}^{N}\frac{\bar{Q}_{ij}^{(k)}(h_k-h_{k-1})}{E_0 h}, \quad D_{ij} = \sum_{k=1}^{N}\frac{\bar{Q}_{ij}^{(k)}}{E_0 h^3}\frac{1}{3}(h_k^3-h_{k-1}^3) \tag{2}$$

and h denotes the total shell thickness, N is the number of laminae, $\rho^{(k)}$ is the density of the kth lamina per unit volume, and ρ_0 and E_0 are a representative material density and a representative elastic modulus, respectively.

By the stationary condition of eqn (1), one obtains the equations of motion as follows:

$$E_1 = \alpha^4 H_1 u + \beta^2\left[A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\left(\frac{\partial^2 v}{\partial x\,\partial \theta}+\frac{\partial w}{\partial x}\right) + A_{66}\left(\frac{\partial^2 v}{\partial x\,\partial \theta}+\frac{\partial^2 u}{\partial \theta^2}\right)\right]$$
$$-D_{11}\frac{\partial^3 w}{\partial x^3} + D_{66}\left(\frac{\partial^3 w}{\partial x\,\partial \theta^2}+\frac{\partial^2 u}{\partial \theta^2}\right) = 0$$

$$E_2 = \alpha^4 H_1 v + \beta^2\left[A_{12}\frac{\partial^2 u}{\partial x\,\partial \theta} + A_{22}\left(\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial w}{\partial \theta}\right) + A_{66}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\,\partial \theta}\right)\right]$$
$$-D_{12}\frac{\partial^3 w}{\partial x^2\,\partial \theta} + 3D_{66}\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^3 w}{\partial x^2\,\partial \theta}\right) = 0$$

$$E_3 = \alpha^4 H_1 w - \beta^2\left[A_{12}\frac{\partial u}{\partial x} + A_{22}\left(\frac{\partial v}{\partial \theta}+w\right)\right] + D_{11}\left(\frac{\partial^3 u}{\partial x^3}-\frac{\partial^4 w}{\partial x^4}\right)$$
$$+D_{12}\left(\frac{\partial^3 v}{\partial x^2\,\partial \theta} - 2\frac{\partial^4 w}{\partial x^2\,\partial \theta^2}\right) - D_{22}\left(\frac{\partial^4 w}{\partial \theta^4}+2\frac{\partial^2 w}{\partial \theta^2}+w\right)$$
$$+D_{66}\left(3\frac{\partial^3 v}{\partial x^2\,\partial \theta} - 4\frac{\partial^4 w}{\partial x^2\,\partial \theta^2} - \frac{\partial^3 u}{\partial x\,\partial \theta^2}\right) = 0 \tag{3}$$

With the solutions of the equations of motion (3) denoted by u, v and w, one can integrate the Lagrangian by parts with the help of the equations for E_1, E_2 and E_3, and, for a complete circular cylindrical shell in the θ direction, can eventually obtain the following Lagrangian expression:

$$-L\Big/\left(\frac{D}{2R_0^2}\right)=\int_0^{2\pi}\left[T_1u+T_2v+T_3w+M_1\frac{\partial w}{\partial x}\right]\Bigg|_{x=-\mu}^{x=\mu}\mathrm{d}\theta \tag{4}$$

where

$$T_1=\beta^2\left[A_{11}\frac{\partial u}{\partial x}+A_{12}\left(\frac{\partial v}{\partial\theta}+w\right)\right]-D_{11}\frac{\partial^2 w}{\partial x^2}$$

$$T_2=\beta^2A_{66}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial\theta}\right)+3D_{66}\left(\frac{\partial v}{\partial x}-\frac{\partial^2 w}{\partial x\,\partial\theta}\right)$$

$$T_3=D_{11}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^3 w}{\partial x^3}\right)+D_{12}\left(\frac{\partial^2 v}{\partial x\,\partial\theta}-\frac{\partial^3 w}{\partial x\,\partial\theta^2}\right)+D_{66}\left(3\frac{\partial^2 v}{\partial x\,\partial\theta}-4\frac{\partial^3 w}{\partial x\,\partial\theta^2}-\frac{\partial^2 u}{\partial\theta^2}\right)$$

$$M_1=D_{11}\left(-\frac{\partial u}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+D_{12}\left(-\frac{\partial v}{\partial\theta}+\frac{\partial^2 w}{\partial\theta^2}\right) \tag{5}$$

With the integration constants denoted as C_{nq}, C^*_{nq} (q = 1–4), the general solutions of equations of motion (3) are expressed as follows (Hu *et al.*, 1996):

$$u=\sum_{n=1}^{\infty}\sum_{q=1}^{4}[C_{nq}U_{nq}(x)+C^*_{nq}U^*_{nq}(x)]\cos(n\theta)$$

$$v=\sum_{n=1}^{\infty}\sum_{q=1}^{4}[C_{nq}V_{nq}(x)+C^*_{nq}V^*_{nq}(x)]\sin(n\theta)$$

$$w=\sum_{n=1}^{\infty}\sum_{q=1}^{4}[C_{nq}W_{nq}(x)+C^*_{nq}W^*_{nq}(x)]\cos(n\theta) \tag{6}$$

Here $(U_{nq}, V_{nq}, W_{nq})_{q=1-4}$ and $(U^*_{nq}, V^*_{nq}, W^*_{nq})_{q=1-4}$ are the solutions in the form of power series expansions, and they are the solutions for the symmetric vibration with respect to the $x = 0$ plane and the antisymmetric one, respectively. The circumferential wave number is denoted by n. One can expand displacements and slopes at the boundary in Fourier series as follows:

$$u|_{x=\pm\mu}=\sum_{n=1}^{\infty}(\pm a_{n1}+a^*_{n1})\cos(n\theta)$$

$$v|_{x=\pm\mu}=\sum_{n=1}^{\infty}(a_{n2}\pm a^*_{n2})\sin(n\theta)$$

$$w|_{x=\pm\mu}=\sum_{n=1}^{\infty}(a_{n3}\pm a^*_{n3})\cos(n\theta)$$

$$\frac{\partial w}{\partial x}\bigg|_{x=\pm\mu}=\sum_{n=1}^{\infty}(\pm a_{n4}+a^*_{n4})\cos(n\theta) \tag{7}$$

The double signs are taken in the same order.

Hereafter, a_{nq}, a^*_{nq} (q = 1–4) are called the boundary values. Substituting eqns (6) into eqns (7) gives integration constants, in which $[U_{n1}, V_{n1}, W_{n1}, \mathrm{d}W_{n1}/\mathrm{d}x]_{x=\mu}$, etc. are abbreviated as U_{n1}, V_{n1}, W_{n1}, W'_{n1}, etc.:

$$C_{nq} = \frac{1}{\Delta_0}\sum_{p=1}^{4}(-1)^{p+q}a_{np}\Delta_{pq}(q = 1\text{–}4) \quad (8)$$

Here

$$\Delta_0 = \begin{vmatrix} U_{n1} & U_{n2} & U_{n3} & U_{n4} \\ V_{n1} & V_{n2} & V_{n3} & V_{n4} \\ W_{n1} & W_{n2} & W_{n3} & W_{n4} \\ W'_{n1} & W'_{n2} & W'_{n3} & W'_{n4} \end{vmatrix} \quad (9)$$

The symbol Δ_{pq} denotes the 3×3 minor determinant obtained by eliminating the pth row and qth column from Δ_0. By replacing $(U_{nq}, V_{nq}, W_{nq}, W'_{nq})_{q=1-4}$ with $(U^*_{nq}, V^*_{nq}, W^*_{ng}, W'^*_{nq})_{q=1-4}$ in Δ_0 and $(\Delta_{pq})_{p,q=1-4}$, one obtains Δ^*_0 and $(\Delta^*_{pq})_{p,q=1-4}$, and furthermore, by replacing a_{nq}, Δ_0, Δ_{pq} with a^*_{np}, Δ^*_0, Δ^*_{pq} in C_{nq} one obtains the expression for C^*_{nq}. Substituting eqns (6) into eqns (5) and putting $x = \pm\mu$ gives the following equations,

$$T_1|_{x=\pm\mu} = \sum_{n=1}^{\infty}\sum_{q=1}^{4}(C_{nq}T_{1q} \pm C^*_{nq}T^*_{1q})\cos(n\theta)$$

$$T_2|_{x=\pm\mu} = \sum_{n=1}^{\infty}\sum_{q=1}^{4}(\pm C_{nq}T_{2q} + C^*_{nq}T^*_{2q})\sin(n\theta)$$

$$T_3|_{x=\pm\mu} = \sum_{n=1}^{\infty}\sum_{q=1}^{4}(\pm C_{nq}T_{3q} + C^*_{nq}T^*_{3q})\cos(n\theta)$$

$$M_1|_{x=\pm\mu} = \sum_{n=1}^{\infty}\sum_{q=1}^{4}(C_{nq}T_{4q} \pm C^*_{nq}T^*_{4q})\cos(n\theta) \quad (10)$$

The double signs are taken in the same order.

Substituting eqns (7), (8) and (10) into eqn (4), gives the Lagrangian expressed in a quadratic form of the boundary values as follows:

$$L = \sum_{n=1}^{\infty}(L_{sn} + L_{an}) \quad (11)$$

where

$$-L_{sn}\Big/\left(\frac{\pi D}{R_0^2}\right) = \frac{1}{\Delta_0}\left[\sum_{p=1}^{4}S_{pp}a_{np}^2 + \sum_{i=1}^{3}\sum_{j=i+1}^{4}S_{ij}a_{ni}a_{nj}\right]$$

$$S_{pp} = \sum_{q=1}^{4}(-1)^{p+q}\Delta_{pq}T_{pq}$$

$$S_{ij} = \sum_{q=1}^{4}\left[(-1)^{q+i}\Delta_{iq}T_{jq} + (-1)^{q+j}\Delta_{jq}T_{iq}\right] \quad (12)$$

One can obtain L_{an} by replacing a_{np}, Δ_0, Δ_{pq} and T_{pq} with a^*_{np}, Δ^*_0, Δ^*_{pq} and T^*_{pq} in L_{sn}.

2.2. *Lagrangian of a laminated cross-ply circular plate*

Let us consider the free vibrations of a laminated cross-ply thin circular plate composed of N laminae symmetric about the middle surface. The outside and the inside radii of the plate are denoted with R_1 and R_2, respectively, and h_a denotes the thickness.

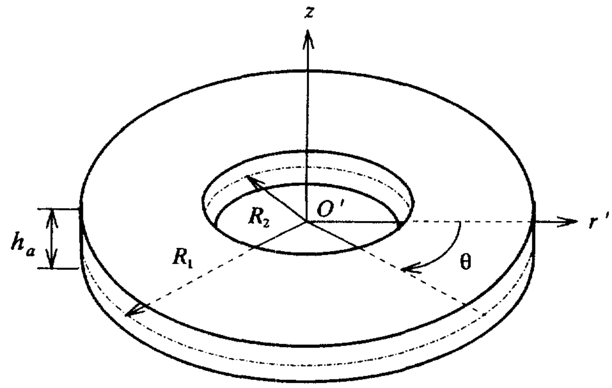

Fig. 3. Coordinate system of a circular plate.

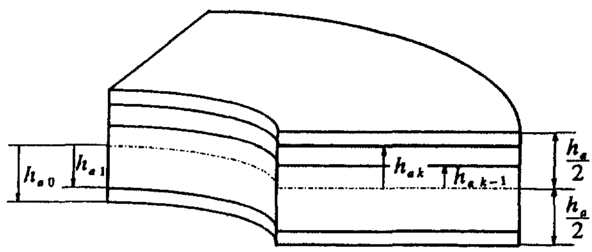

Fig. 4. Cross-sectional view of a circular plate.

As shown in Fig. 3, r' and θ are taken to be the polar co-ordinates on the middle surface of the plate and the z-axis is taken to be perpendicular to the middle surface. Figure 4 shows a cross-sectional view of the plate, in which h_{ak} and h_{ak-1} are the distances measured from the middle surface to the outer and inner surfaces of the kth lamina, respectively. The present analysis is limited to plates sufficiently thin, so that the effects of shear deformation and rotary inertia may be neglected. The displacements in the r', θ and z directions may be assumed in the form

$$\begin{aligned}\bar{u} &= \bar{u}_a - z\frac{\partial \bar{w}_a}{\partial r'}\\ \bar{v} &= \bar{v}_a - \frac{z}{r'}\frac{\partial \bar{w}_a}{\partial \theta}\\ \bar{w} &= \bar{w}_a\end{aligned} \tag{13}$$

where u_a, v_a, w_a are functions of r', θ and t (time).

The normal and the shearing strain expressions at any point are obtained from Love (1927) and are given as follows:

$$\begin{aligned}\varepsilon_{r'} &= \frac{\partial \bar{u}_a}{\partial r'} - z\frac{\partial^2 \bar{w}_a}{\partial r'^2}\\ \varepsilon_\theta &= \frac{\bar{u}_a}{r'} + \frac{1}{r'}\frac{\partial \bar{v}_a}{\partial \theta} - \frac{z}{r'}\left(\frac{\partial \bar{w}_a}{\partial r'} + \frac{1}{r'}\frac{\partial^2 \bar{w}_a}{\partial \theta^2}\right)\\ \varepsilon_z &= 0\\ \varepsilon_{r'\theta} &= \frac{1}{r'}\frac{\partial \bar{u}_a}{\partial \theta} + \frac{\partial \bar{v}_a}{\partial r'} - \frac{\bar{v}_a}{r'} + \frac{2z}{r'}\left(\frac{1}{r'}\frac{\partial \bar{w}_a}{\partial \theta} - \frac{\partial^2 \bar{w}_a}{\partial r'\,\partial \theta}\right)\end{aligned} \tag{14}$$

Consider now that each lamina may be regarded on a macroscopic scale as being homogeneous and orthotropic. The stress–strain relation for the kth lamina may be written as (cf Whitney, 1987; Vinson and Sierakowski, 1987)

$$\begin{Bmatrix} \sigma_{r'} \\ \sigma_\theta \\ \sigma_{r'\theta} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix} \begin{bmatrix} \varepsilon_{r'} \\ \varepsilon_\theta \\ \varepsilon_{r'\theta} \end{bmatrix}_k \tag{15}$$

where $\sigma_{r'}$ and σ_θ are normal stress components, and $\sigma_{r'\theta}$ is a shear stress component. The constants $\bar{Q}_{ij}$ are the elastic stiffness coefficients for the material. The strain energy for the kth lamina is given as

$$U_{ak} = \frac{1}{2}\int_{ha_{k-1}}^{ha_k}\int_0^{2\pi}\int_{R_2}^{R_1}[\sigma_{r'}\varepsilon_{r'}+\sigma_\theta\varepsilon_\theta+\sigma_{r'\theta}\varepsilon_{r'\theta}]r'\,\mathrm{d}r'\,\mathrm{d}\theta\,\mathrm{d}z \tag{16}$$

The kinetic energy for each lamina is given as

$$T_{ak} = \frac{1}{2}\int\int\int\rho_a^{(k)}\left[\left(\frac{\partial\bar{u}_a}{\partial t}\right)^2+\left(\frac{\partial\bar{v}_a}{\partial t}\right)^2+\left(\frac{\partial\bar{w}_a}{\partial t}\right)^2\right]r'\,\mathrm{d}r'\,\mathrm{d}\theta\,\mathrm{d}z \tag{17}$$

where $\rho_a^{(k)}$ is the density of the kth lamina per unit volume.

Now define the Lagrangian in a period (τ) of vibration as follows:

$$L_{ak} = \frac{2}{\tau}\int_0^\tau (T_{ak}-U_{ak})\,\mathrm{d}t \tag{18}$$

By assuming that $\{\bar{u}_a, \bar{v}_a, \bar{w}_a\} = \{u_a(r',\theta), v_a(r',\theta), w_a(r',\theta)\}\sin(pt)$, the Lagrangian of the entire plate is then found to be

$$L = \sum_{k=1}^{N} L_{ak} \tag{19}$$

where N is the total number of laminae and p is the circular frequency. Employing a nondimensional coordinate $r = \alpha_1 r'/R_1$ and substituting eqns (13)–(15) into eqns (16)–(19), eqn (19) becomes

$$L = L_B + L_I \tag{20}$$

where

$$-L_B\Big/\left(\frac{D_a\alpha_1^2}{2R_1^2}\right) = \int_0^{2\pi}\int_{\alpha_2}^{\alpha_1}\left\{-H_{a1}w_a^2+D_{a11}\left(\frac{\partial^2 w_a}{\partial r^2}\right)^2+2D_{a12}\frac{\partial^2 w_a}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_a}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_a}{\partial\theta^2}\right)\right.$$
$$\left.+D_{a22}\left(\frac{1}{r}\frac{\partial w_a}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_a}{\partial\theta^2}\right)^2+4D_{a66}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_a}{\partial\theta}\right)\right]^2\right\}r\,\mathrm{d}r\,\mathrm{d}\theta \tag{21}$$

$$-L_I\Big/\left(\frac{D_a\alpha_1^2}{2R_1^2}\right) = \int_0^{2\pi}\int_{\alpha_2}^{\alpha_1}\left\{-H_{a1}(u_a^2+v_a^2)+\frac{1}{\kappa^2}\left[A_{a11}\left(\frac{\partial u_a}{\partial r}\right)^2+2A_{a12}\frac{\partial u_a}{\partial r}\left(\frac{u_a}{r}+\frac{1}{r}\frac{\partial v_a}{\partial\theta}\right)\right.\right.$$
$$\left.\left.+A_{a22}\left(\frac{u_a}{r}+\frac{1}{r}\frac{\partial v_a}{\partial\theta}\right)^2+A_{a66}\left(\frac{1}{r}\frac{\partial u_a}{\partial\theta}+\frac{\partial v_a}{\partial r}-\frac{v_a}{r}\right)^2\right]\right\}r\,\mathrm{d}r\,\mathrm{d}\theta \tag{22}$$

$$D_a = E_{a0} h_a^3$$

$$\beta_a = R_1/h_a$$

$$\alpha_1^4 = \rho_{a0} h_a p^2 R_1^4/D_a$$

$$\alpha_2^4 = \rho_{a0} h_a p^2 R_2^4/D_a$$

$$\kappa = \alpha_1/\beta_a$$

$$H_{a1} = \sum_{k=1}^{N} \frac{\rho_a^{(k)}(h_{ak}-h_{ak-1})}{\rho_{a0} h_a}$$

$$A_{aij} = \sum_{k=1}^{N} \frac{\bar{Q}_{ij}^{(k)}(h_{ak}-h_{ak-1})}{E_{a0} h_a}$$

$$D_{aij} = \sum_{k=1}^{N} \frac{\bar{Q}_{ij}^{(k)}}{E_{a0} h_a^3} \frac{1}{3}(h_{ak}^3 - h_{ak-1}^3) \tag{23}$$

Moreover, h_a is the total plate thickness, N is the number of laminae, ρ_{a0} and E_{a0} are a representative material density and a representative elastic modulus, respectively. Equation (21) corresponds to the Lagrangian for out-of-plane vibrations and eqn (22) to that for in-plane vibrations.

2.2.1. *Lagrangian for out-of-plane vibrations in terms of boundary values.* By the stationary condition of eqn (21), one obtains the equation of motion as follows:

$$\begin{aligned} E_a = -H_{a1} w_a &+ \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\left[D_{a11}\frac{\partial^2 w_a}{\partial r^2} + D_{a12}\left(\frac{1}{r}\frac{\partial w_a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_a}{\partial r^2}\right)\right] \\ &+ \left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\left[D_{a22}\left(\frac{1}{r}\frac{\partial w_a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_a}{\partial \theta^2}\right) + D_{a12}\frac{\partial^2 w_a}{\partial r^2}\right] \\ &+ 4D_{a66}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\right)\left[\frac{\partial^2}{\partial r \partial \theta}\left(\frac{1}{r}\frac{\partial w_a}{\partial \theta}\right)\right] = 0 \end{aligned} \tag{24}$$

With the solutions (see Appendix A) of the equation of motion (24), one can integrate the Lagrangian by parts, and for a complete circular plate in the θ direction, can eventually obtain the following Lagrangian expression:

$$-L_B \Big/ \left(\frac{D_a \alpha_1^2}{2R_1^2}\right) = \int_0^{2\pi} \left[T_{a1} w_a + M_{a1}\frac{\partial w_a}{\partial r}\right]\Bigg|_{r=\alpha_2}^{r=\alpha_1} \mathrm{d}\theta \tag{25}$$

where

$$\begin{aligned} T_{a1} = D_{a22}\left(\frac{1}{r}\frac{\partial w_a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_a}{\partial \theta^2}\right) + D_{a12}\frac{\partial^2 w_a}{\partial r^2} - \frac{\partial}{\partial r}\left\{r\left[D_{a11}\frac{\partial^2 w_a}{\partial r^2}\right.\right. \\ \left.\left. + D_{a12}\left(\frac{1}{r}\frac{\partial w_a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_a}{\partial \theta^2}\right)\right]\right\} - 4D_a\frac{1}{r}\left(-\frac{1}{r}\frac{\partial^2 w_a}{\partial \theta^2} + \frac{\partial^3 w_a}{\partial r \partial \theta^2}\right) \end{aligned}$$

$$M_{a1} = r\left[D_{a11}\frac{\partial^2 w_a}{\partial r^2} + D_{a12}\left(\frac{1}{r}\frac{\partial w_a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_a}{\partial \theta^2}\right)\right] \tag{26}$$

With the integration constants denoted as b_{nq} ($q = 1$–4), the general solutions of equation of motion (24) are expressed as follows:

$$w_a = \sum_{n=1}^{\infty} \sum_{q=1}^{4} [b_{nq} w_{nq}(r)] \cos(n\theta) \tag{27}$$

Here w_{nq} ($q = 1$–4) are the solutions in the form of power series expansions. One can expand displacements and slopes at the boundary in Fourier series as follows:

$$\begin{aligned} w_a|_{r=\alpha_1} &= \sum_{n=1}^{\infty} d_{n1} \cos(n\theta) \\ \left.\frac{\partial w_a}{\partial r}\right|_{r=\alpha_1} &= \sum_{n=1}^{\infty} d_{n2} \cos(n\theta) \\ w_a|_{r=\alpha_2} &= \sum_{n=1}^{\infty} d_{n3} \cos(n\theta) \\ \left.\frac{\partial w_a}{\partial r}\right|_{r=\alpha_2} &= \sum_{n=1}^{\infty} d_{n4} \cos(n\theta) \end{aligned} \tag{28}$$

Hereafter, d_{nq} are called the boundary values. Substituting eqn (27) into eqns (28) gives integration constants, in which $\mathrm{d}w_{nq}/\mathrm{d}r$ is abbreviated as w'_{nq}:

$$b_{nq} = \frac{1}{\Delta_{B0}} \sum_{p=1}^{4} (-1)^{p+q} d_{np} \Delta_{Bpq} \quad (q = 1\text{–}4) \tag{29}$$

where

$$\Delta_{B0} = \begin{vmatrix} w_{n1(\alpha_1)} & w_{n2(\alpha_1)} & w_{n3(\alpha_1)} & w_{n4(\alpha_1)} \\ w'_{n1(\alpha_1)} & w'_{n2(\alpha_1)} & w'_{n3(\alpha_1)} & w'_{n4(\alpha_1)} \\ w_{n1(\alpha_2)} & w_{n2(\alpha_2)} & w_{n3(\alpha_2)} & w_{n4(\alpha_2)} \\ w'_{n1(\alpha_2)} & w'_{n2(\alpha_2)} & w'_{n3(\alpha_2)} & w'_{n4(\alpha_2)} \end{vmatrix} \tag{30}$$

The symbol Δ_{Bpq} denotes the 3×3 determinant obtained by eliminating the pth row and qth column from Δ_{B0}.

Substituting eqn (27) into eqns (26) and putting $r = \alpha_1$ and $r = \alpha_2$ gives the following equations:

$$\begin{aligned} T_{a1}|_{r=\alpha_1} = T_{B1} &= \sum_{n=1}^{\infty} \sum_{q=1}^{4} (b_{nq} T_{B1q}) \cos(n\theta) \\ M_{a1}|_{r=\alpha_1} = T_{B2} &= \sum_{n=1}^{\infty} \sum_{q=1}^{4} (b_{nq} T_{B2q}) \cos(n\theta) \\ T_{a1}|_{r=\alpha_2} = T_{B3} &= \sum_{n=1}^{\infty} \sum_{q=1}^{4} (b_{nq} T_{B3q}) \cos(n\theta) \\ M_{a1}|_{r=\alpha_2} = T_{B4} &= \sum_{n=1}^{\infty} \sum_{q=1}^{4} (b_{nq} T_{B4q}) \cos(n\theta) \end{aligned} \tag{31}$$

Substituting eqns (28), (29) and (31) into eqn (25) gives the Lagrangian expressed in a quadratic form of the boundary values as follows:

$$L_B = \sum_{n=1}^{\infty} L_{Bn} \tag{32}$$

where

$$-L_{Bn}\Bigg/\left(\frac{\pi D_a \alpha_1^2}{2R_1^2}\right) = \frac{1}{\Delta_{B0}}\left[\sum_{p=1}^{4}(S_{Bpp}d_{np}^2) + \sum_{i=1}^{3}\sum_{j=i+1}^{4}(S_{Bij}d_{ni}d_{nj})\right] \tag{33}$$

$$S_{Bpp} = \sum_{q=1}^{4}[(-1)^{p+q}\Delta_{Bpq}T_{Bpq}]$$

$$S_{Bij} = \sum_{q=1}^{4}[(-1)^{q+i}\Delta_{Biq}T_{Bjq} + (-1)^{q+j}\Delta_{Bjq}T_{Biq}] \tag{34}$$

2.2.2. *Lagrangian for in-plane vibrations in terms of boundary values.* By the stationary condition of eqn (22), one obtains the equations of motion as follows:

$$E_{b1} = H_{a1}\kappa^2 u_a r + A_{a11}\left(\frac{\partial u_a}{\partial r} + r\frac{\partial^2 u_a}{\partial r^2}\right) + A_{a12}\frac{\partial^2 v_a}{\partial r\,\partial\theta} - A_{a22}\left(\frac{1}{r}u_a + \frac{1}{r}\frac{\partial v_a}{\partial\theta}\right)$$

$$+A_{a66}\left(\frac{1}{r}\frac{\partial^2 u_a}{\partial\theta^2} + \frac{\partial^2 v_a}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial v_a}{\partial\theta}\right) = 0$$

$$E_{b2} = H_{a1}\kappa^2 v_a r + A_{a22}\left(\frac{1}{r}\frac{\partial u_a}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v_a}{\partial\theta^2}\right) + A_{a12}\frac{\partial^2 u_a}{\partial r\,\partial\theta}$$

$$+A_{a66}\left(\frac{1}{r}\frac{\partial u_a}{\partial\theta} + \frac{\partial^2 u_a}{\partial r\,\partial\theta} + \frac{\partial v_a}{\partial r} + r\frac{\partial^2 v_a}{\partial r^2} - \frac{v_a}{r}\right) = 0 \tag{35}$$

With the solutions (see Appendix A) of the equations of motion (35), one can integrate the Lagrangian (22) by parts and obtain the following Lagrangian expression:

$$-L_I\Bigg/\left(\frac{D_a\alpha_1^2}{2R_1^2}\right) = \int_0^{2\pi}[T_{b1}u_a + T_{b2}v_a]|_{r=\alpha_2}^{r=\alpha_1}\,d\theta \tag{36}$$

where

$$T_{b1} = \left[A_{a11}\left(\frac{\partial u_a}{\partial r}\right)r + A_{a12}\left(u_a + \frac{\partial v_a}{\partial\theta}\right)\right]\Bigg/\kappa^2$$

$$T_{b2} = \left[A_{a66}\left(\frac{\partial u_a}{\partial\theta} + r\frac{\partial v_a}{\partial r} - v_a\right)\right]\Bigg/\kappa^2 \tag{37}$$

With the integration constants denoted as e_{nq} (q = 1–4), the general solutions of equations of motion (35) are expressed as follows:

$$u_a = \sum_{n=1}^{\infty}\sum_{q=1}^{4}(e_{nq}u_{nq})\cos(n\theta)$$

$$v_a = \sum_{n=1}^{\infty}\sum_{q=1}^{4}(e_{nq}v_{nq})\sin(n\theta) \tag{38}$$

Here u_{nq} and v_{nq} are the solutions in the form of power series expansions. One can expand displacements and slopes at the boundary in Fourier series as follows:

$$u_a|_{r=\alpha_1} = \sum_{n=1}^{\infty} f_{n1} \cos(n\theta)$$

$$v_a|_{r=\alpha_1} = \sum_{n=1}^{\infty} f_{n2} \sin(n\theta)$$

$$u_a|_{r=\alpha_2} = \sum_{n=1}^{\infty} f_{n3} \cos(n\theta)$$

$$v_a|_{r=\alpha_2} = \sum_{n=1}^{\infty} f_{n4} \sin(n\theta) \tag{39}$$

Hereafter f_{nq} are called the boundary values.

Substituting eqns (38) into eqns (39) gives integration constants:

$$e_{nq} = \frac{1}{\Delta_{I0}} \sum_{p=1}^{4} (-1)^{p+q} f_{np} \Delta_{Ipq} \quad (q = 1\text{–}4) \tag{40}$$

where

$$\Delta_{I0} = \begin{vmatrix} u_{n1(\alpha_1)} & u_{n2(\alpha_1)} & u_{n3(\alpha_1)} & u_{n4(\alpha_1)} \\ v_{n1(\alpha_1)} & v_{n2(\alpha_1)} & v_{n3(\alpha_1)} & v_{n4(\alpha_1)} \\ u_{n1(\alpha_2)} & u_{n2(\alpha_2)} & u_{n3(\alpha_2)} & u_{n4(\alpha_2)} \\ v_{n1(\alpha_2)} & v_{n2(\alpha_2)} & v_{n3(\alpha_2)} & v_{n4(\alpha_2)} \end{vmatrix} \tag{41}$$

The symbol Δ_{Ipq} denotes the 3×3 determinant obtained by eliminating the pth row and qth column from Δ_{I0}.

Substituting equations (38) into eqns (37) and putting $r = \alpha_1, \alpha_2$ gives the following equations:

$$T_{b1}|_{r=\alpha_1} = T_{I1} = \sum_{n=1}^{\infty} \sum_{q=1}^{4} (e_{nq} T_{I1q}) \cos(n\theta)$$

$$T_{b2}|_{r=\alpha_1} = T_{I2} = \sum_{n=1}^{\infty} \sum_{q=1}^{4} (e_{nq} T_{I2q}) \sin(n\theta)$$

$$T_{b1}|_{r=\alpha_2} = T_{I3} = \sum_{n=1}^{\infty} \sum_{q=1}^{4} (e_{nq} T_{I3q}) \cos(n\theta)$$

$$T_{b2}|_{r=\alpha_2} = T_{I4} = \sum_{n=1}^{\infty} \sum_{q=1}^{4} (e_{nq} T_{I4q}) \sin(n\theta) \tag{42}$$

Substituting eqns (39), (40) and (42) into eqn (36) gives

$$L_I = \sum_{n=1}^{\infty} L_{In} \tag{43}$$

where

$$-L_{In} \Big/ \left(\frac{\pi D_a \alpha_1^2}{2R_1^2} \right) = \frac{1}{\Delta_{I0}} \left[\sum_{p=1}^{4} (S_{Ipp} f_{np}^2) + \sum_{i=1}^{3} \sum_{j=i+1}^{4} (S_{Iij} f_{ni} f_{nj}) \right]$$

$$S_{Ipp} = \sum_{q=1}^{4} [(-1)^{p+q} \Delta_{Ipq} T_{Ipq}] \tag{44}$$

$$S_{Iij} = \sum_{q=1}^{4} [(-1)^{q+i}\Delta_{Iiq}T_{Ijq} + (-1)^{q+j}\Delta_{Ijq}T_{Iiq}] \tag{45}$$

2.3. *Lagrangian of a vessel and frequency equation*

Consider the vibrations of a vessel consisting of a circular cylindrical shell and circular plate lids at both ends.

From Fig. 5, the conditions of continuity of the circular cylindrical shell and the circular plate are the same at the upper end and the lower one, namely:

$$\begin{aligned} u_a|_{r=\alpha_1} &= w|_{x=\pm\mu} \\ v_a|_{r=\alpha_1} &= v|_{x=\pm\mu} \\ w_a|_{r=\alpha_1} &= u|_{x=\pm\mu} \\ \left.\frac{\partial w_a}{\partial r}\right|_{r=\alpha_1} &= \left.\frac{1}{\alpha_1}\frac{\partial w}{\partial x}\right|_{x=\pm\mu} \end{aligned} \tag{46}$$

By utilizing eqns (7), (28), (39) and (46), the following relations among the boundary values are obtained:

$$\begin{aligned} d_{n1}|_{x=\pm\mu} &= \pm a_{n1} + a^*_{n1} \\ d_{n2}|_{x=\pm\mu} &= \frac{1}{\alpha_1}(\pm a_{n4} + a^*_{n4}) \\ f_{n1}|_{x=\pm\mu} &= a_{n3} \pm a^*_{n3} \\ f_{n2}|_{x=\pm\mu} &= a_{n2} \pm a^*_{n2} \end{aligned} \tag{47}$$

Moreover, by putting (double signs in the same order)

$$\begin{aligned} d_{np}|_{x=\pm\mu} &= \pm d'_{np} + d^{*\prime}_{np} \\ f_{np}|_{x=\pm\mu} &= f'_{np} \pm f^{*\prime}_{np} \quad (p = 1\text{–}2) \end{aligned} \tag{48}$$

one finds from eqns (47) and (48)

$$\begin{aligned} d'_{n1} &= a_{n1}, \quad d'_{n2} = \frac{1}{\alpha_1} a_{n4} \\ f'_{n1} &= a_{n3}, \quad f'_{n2} = a_{n2} \end{aligned}$$

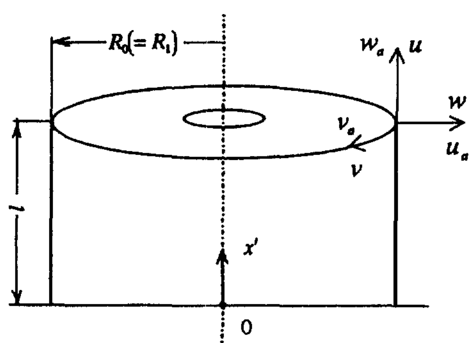

Fig. 5. Conditions of continuity between a circular plate and a cylindrical shell.

$$d_{n1}^{*\prime} = a_{n1}^{*}, \quad d_{n2}^{*\prime} = \frac{1}{\alpha_1} a_{n4}^{*}$$
$$f_{n1}^{*\prime} = a_{n3}^{*}, \quad f_{n2}^{*\prime} = a_{n2}^{*} \tag{49}$$

The Lagrangian of the vessel is the sum of eqn (11) and eqn (20). Upon utilizing eqns (12), (33), (44), (48) and (49), the Lagrangian of the vessel is found to be given as

$$L = L_{\text{symm}} + L_{\text{antisymm}} \tag{50}$$

where L_{symm} is expressed in a quadratic form of boundary values $(a_{n1}, a_{n2}, a_{n3}, a_{n4}, d'_{n3}, d'_{n4}, f'_{n3}, f'_{n4})$ and L_{antisymm} of $(a_{n1}^{*}, a_{n2}^{*}, a_{n3}^{*}, a_{n4}^{*}, d_{n3}^{*\prime}, d_{n4}^{*\prime}, f_{n3}^{*\prime}, f_{n4}^{*\prime})$. The frequency equations of the vessel are obtained from the minimum conditions of the Lagrangian with respect to unknown boundary values.

The frequency equation for the symmetric vibration with respect to the $x = 0$ plane of the vessel is obtained by

$$\partial L/\partial(a_{np}, d'_{nq}, f'_{nq}) = 0$$
$$(a_{np}^{*} = d_{nq}^{*\prime} = f_{nq}^{*\prime} = 0), \quad p = 1\text{–}4, \quad q = 3\text{–}4 \tag{51}$$

The frequency equation for the antisymmetric vibration is obtained by

$$\partial L/\partial(a_{np}^{*}, d_{nq}^{*\prime}, f_{nq}^{*\prime}) = 0$$
$$(a_{np} = d'_{nq} = f'_{nq} = 0), \quad p = 1\text{–}4, \quad q = 3\text{–}4 \tag{52}$$

If the inner edges are clamped, one can put in eqns (51) and (52) $d'_{nq} = f'_{nq} = d_{nq}^{*\prime} = f_{nq}^{*\prime} = 0$. Then the frequency equation can be obtained by the 4×4 determinant of the coefficients a_{np} or a_{np}^{*} $(p = 1\text{–}4)$.

3. NUMERICAL EXAMPLES

Numerical studies were made for vessels of symmetric cross-ply laminates as shown in Fig. 6. As composite materials, graphite fiber reinforced epoxy are considered. Both the number of layers N of the cylinder and the circular plate are three and all the layers are

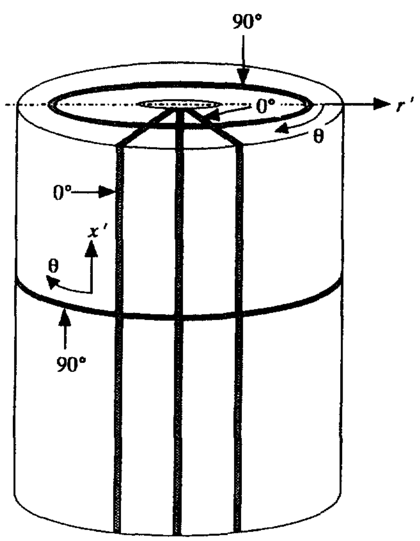

Fig. 6. Analytical model and fiber direction of a vessel.

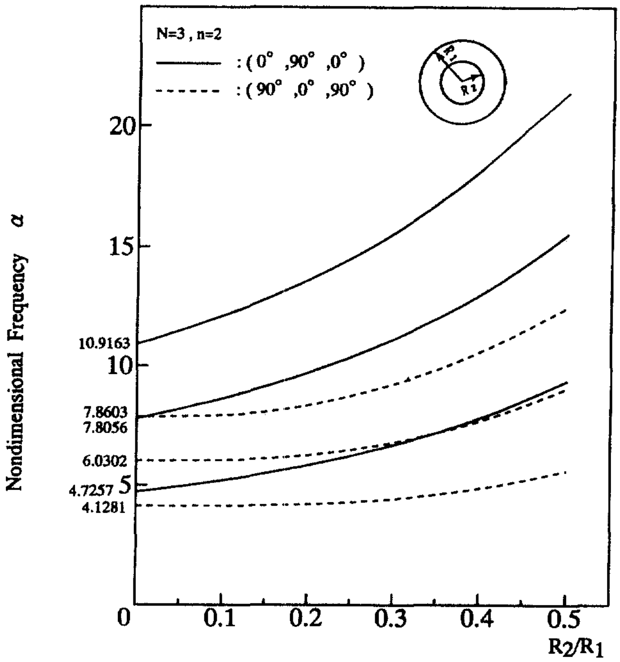

Fig. 7. Frequency curves of a circular plate (inner–outer edges clamped: bending vibration).

taken to have equal thicknesses. The moduli of elasticity of the materials used are taken from Vinson and Sierakowski (1987): $E_{11} = 138$ (GPa), $E_{22} = 8.96$ (GPa), $G_{12} = 7.1$ (GPa), $\nu_{12} = 0.30$. Here E_{11} is the modulus of elasticity of the lamina in the direction of the fibers and E_{22} is the transverse modulus; G_{12} is the shear modulus; and ν_{12} is the Poisson's ratio. When the fibers are directed to the radial direction (r' direction) in the case of the circular plate and to the axial direction (x' direction) in the cylinder, the angle of fibers θ is called zero degree, on the contrary, θ is called 90° when the fibers are directed to the circumferential direction (θ direction), (see Fig. 6).

To confirm the validity of the present method, some results for laminated composite circular cylindrical shells and isotropic circular plates were compared using both the present method and previously published methods. A comparison of the results shows close agreement (see Appendix B).

Figure 7 shows the frequency curves for bending vibration of a circular plate with inner and outer edges clamped and Fig. 8 shows those of a circular plate with inner edge clamped and outer edge free. In the figures, a nondimensional frequency parameter α is used, where

$$\begin{aligned} \alpha^4(=\alpha_1^4) &= \rho h p^2 R_1^4/D \\ D(=D_a) &= Eh^3 \\ E(=E_{a0}) &= E_{11}/[12(1-\nu_{12}\nu_{21})] \end{aligned} \tag{53}$$

and $\rho(=\rho_a^{(k)} = \rho_{a0})$ and $h(=h_a)$ are the density of each lamina and the total plate thickness. The values of α on the ordinate denote the ones when $R_2/R_1 = 10^{-6}$ (Exact solutions of eqn (24) for a circular plate without a hole have not been found yet).

Figures 9–16 show the $\sqrt{\alpha^4/\beta^2}-\mu$ curves of a vessel where the number of laminae $N = 3$ and $R_2/R_1 = 0.1$. In the figures,

$$\begin{aligned} \alpha^4 &= \rho h p^2 R_1^4/D \\ \beta &= R_1/h, \quad \mu = l/R_1 \end{aligned} \tag{54}$$

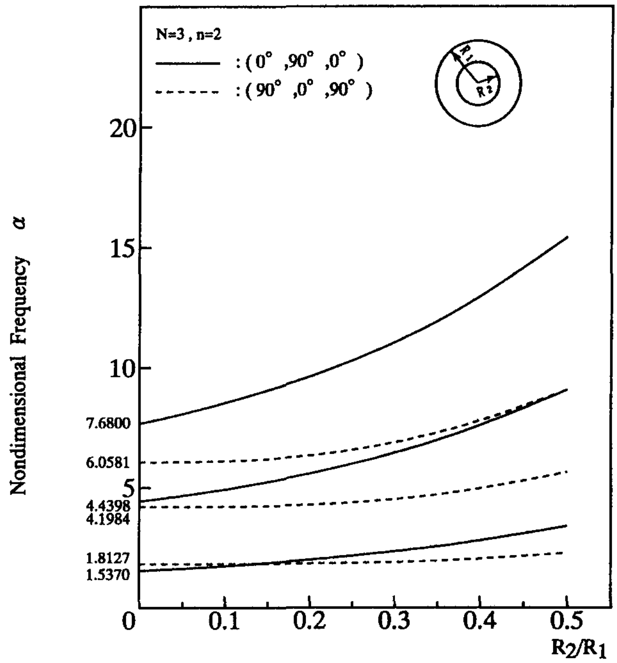

Fig. 8. Frequency curves of a circular plate (inner edge clamped–outer edge free : bending vibration).

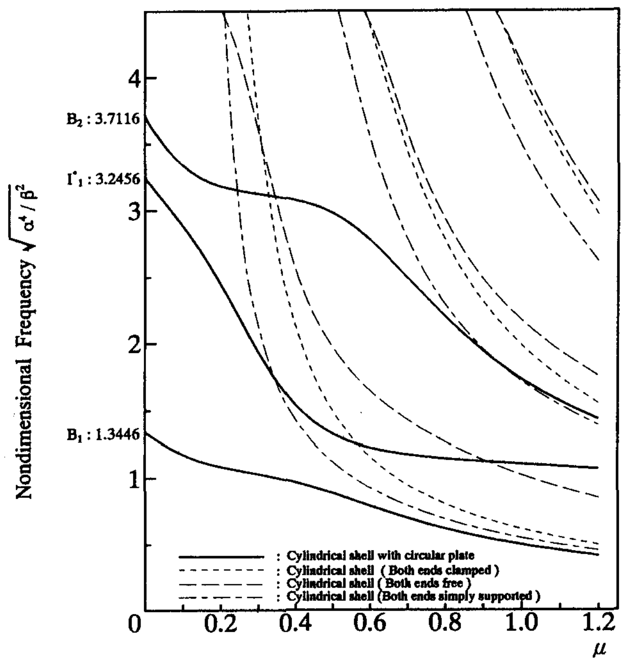

Fig. 9. Frequency curves $(0°, 90°, 0°)$, $n = 2$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

and $(2\mu R_1)$ denotes the total length of the circular cylinder (i.e. the vessel). The values of $\sqrt{\alpha^4/\beta^2}$ at $\mu = 0$ correspond to the eigenvalues for laminated cross-ply circular plates. The symbol B denotes a mode of out-of-plane (bending) vibration, the symbol I denotes one of in-plane vibration, B_i denotes the modes of the circular plate with inner and outer edges clamped, B_i^* the modes of the circular plate with inner edge clamped and outer edge free, B_i' the modes of the circular plate with inner edge simply supported and outer edge clamped and I_i^* the modes of the circular plate with inner edge clamped and outer edge free. As seen

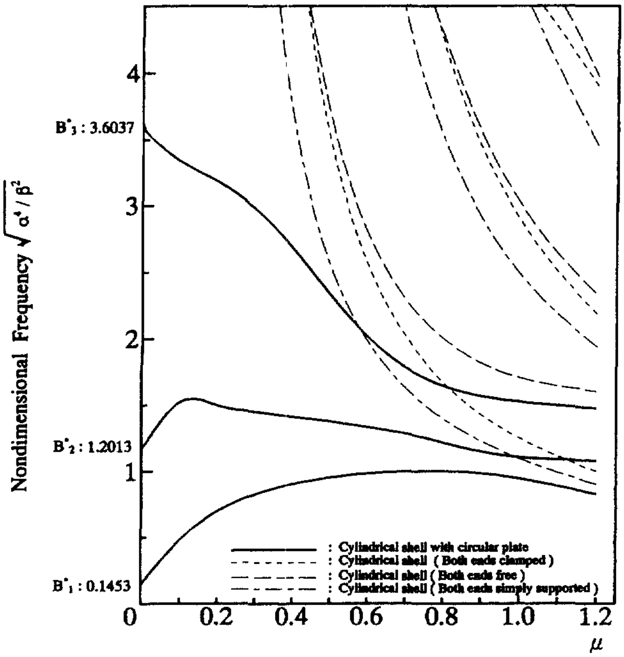

Fig. 10. Frequency curves (0°, 90°, 0°), $n = 2$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: antisymmetric vibration).

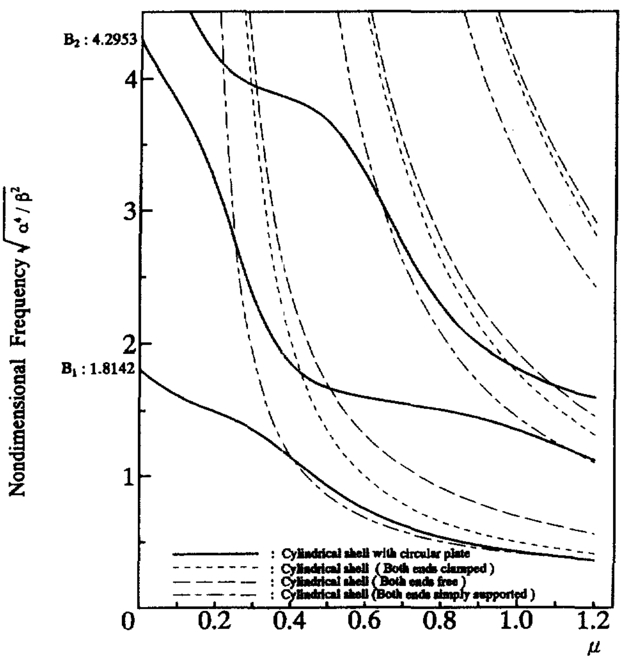

Fig. 11. Frequency curves (0°, 90°, 0°), $n = 4$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

from the figures, the frequency curves of the vessel gradually approach the ones of a circular cylinder with both ends clamped as the length of the cylinder becomes longer. And the frequency curves approach the eigenvalues of circular plates as the lenth of the circular cylinder becomes shorter. The frequencies for the stacking sequence (0°, 90°, 0°) are generally higher than those for (90°, 0°, 90°). The frequency curves vary wavily and complicately with the length of the cylinder.

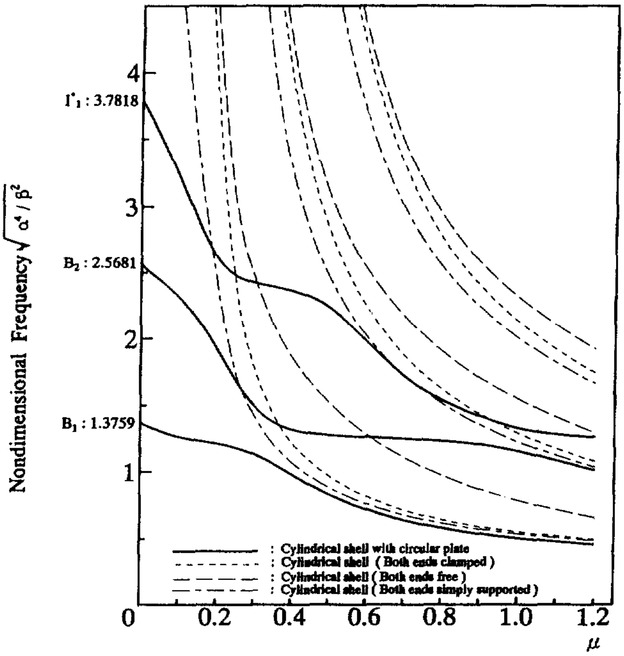

Fig. 12. Frequency curves (90°, 0°, 90°), $n = 3$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

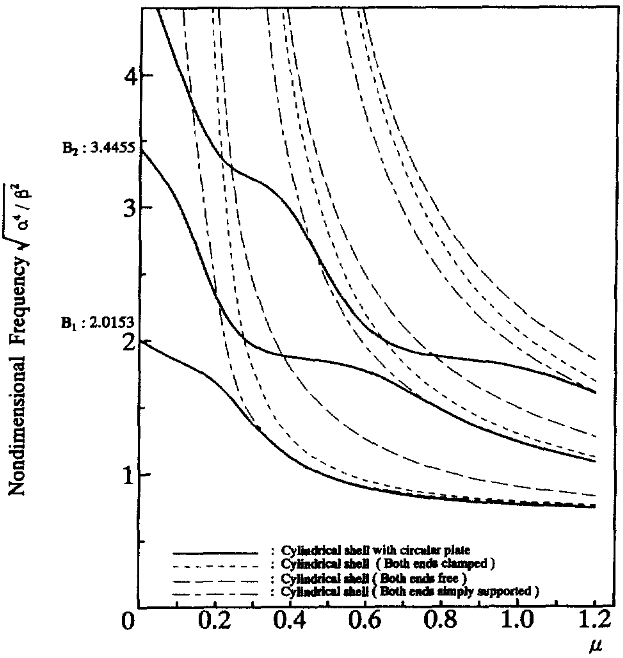

Fig. 13. Frequency curves (90°, 0°, 90°), $n = 4$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

Figure 18 denotes the mode shapes and Fig. 19 does the bending moment distributions for the symmetric vibration of the vessel corresponding to the points S1–S9 on the frequency curves in Fig. 17, in which both the maximum displacement and the maximum bending moment are taken to be unity. Here,

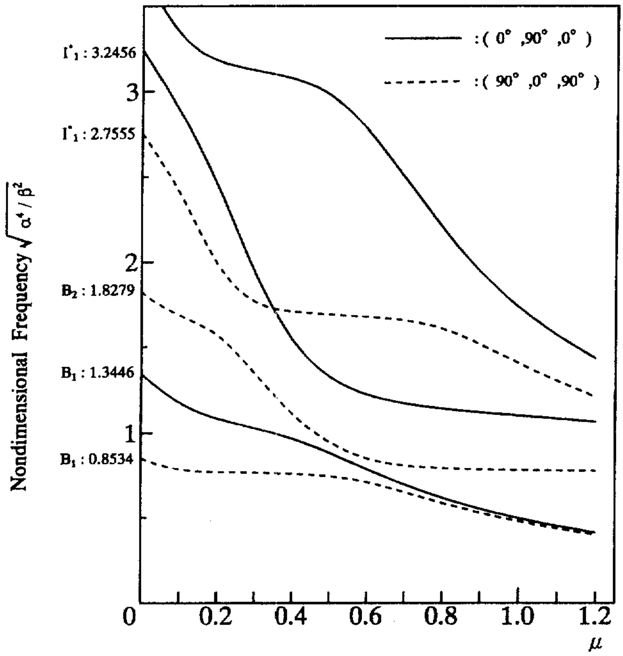

Fig. 14. Frequency curves $n = 2$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

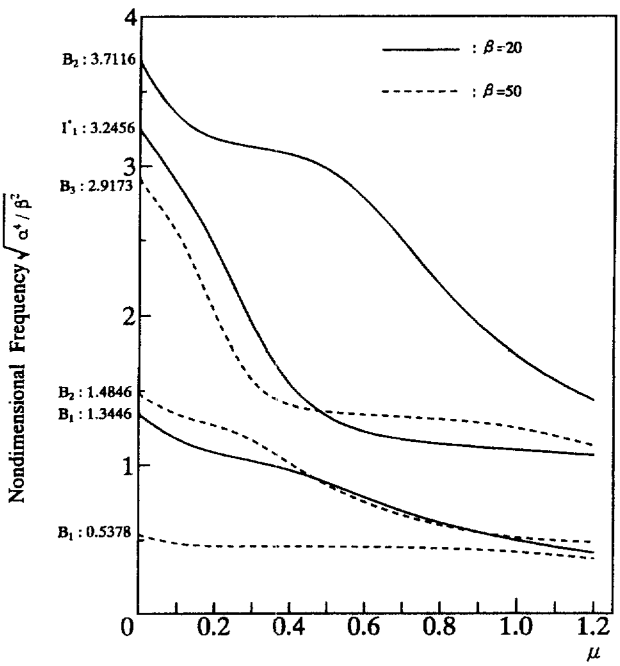

Fig. 15. Frequency curves $(0^\circ, 90^\circ, 0^\circ)$, $n = 2$, $N = 3$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

$$M_x = \frac{D}{R_0^2}\left[D_{11}\left(\frac{\partial u}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + D_{12}\left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right)\right]$$

$$M_r = -\frac{D_a \alpha_1^2}{R_1^2}\left[D_{a11}\frac{\partial^2 w_a}{\partial r^2} + D_{a12}\left(\frac{1}{r}\frac{\partial w_a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_a}{\partial \theta^2}\right)\right] \quad (55)$$

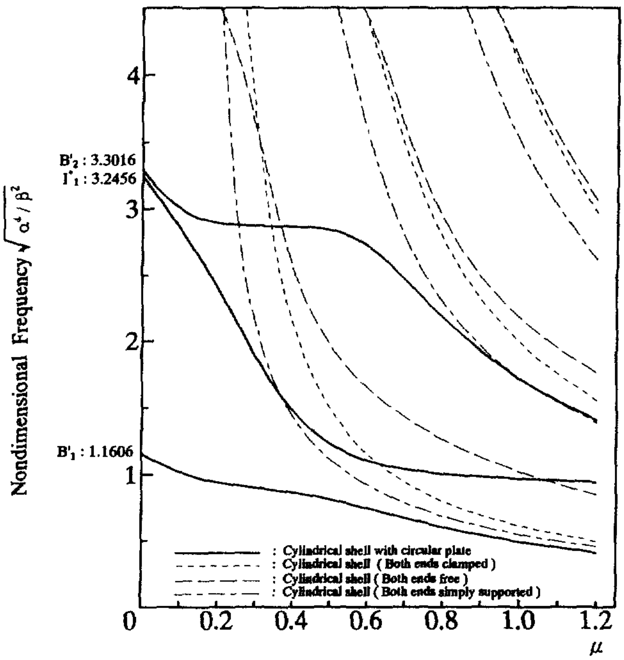

Fig. 16. Frequency curves $(0°, 90°, 0°)$, $n = 2$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges simply supported: symmetric vibration).

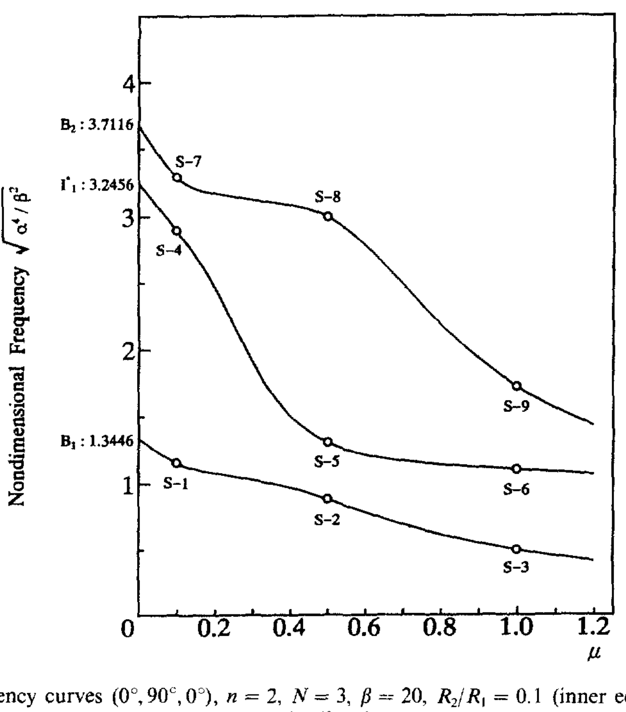

Fig. 17. Frequency curves $(0°, 90°, 0°)$, $n = 2$, $N = 3$, $\beta = 20$, $R_2/R_1 = 0.1$ (inner edges clamped: symmetric vibration).

One can see from the figures that even the mode shapes that correspond to the points on the frequency curve for the same mode of vibration vary with the length of the circular cylinder. The vibration of the circular plate is dominant when the cylinder is short and the mode shapes of the plate are complicated but those of the cylinder are simple. On the contrary, the mode shapes of the cylinder are complicated but those of the circular plate

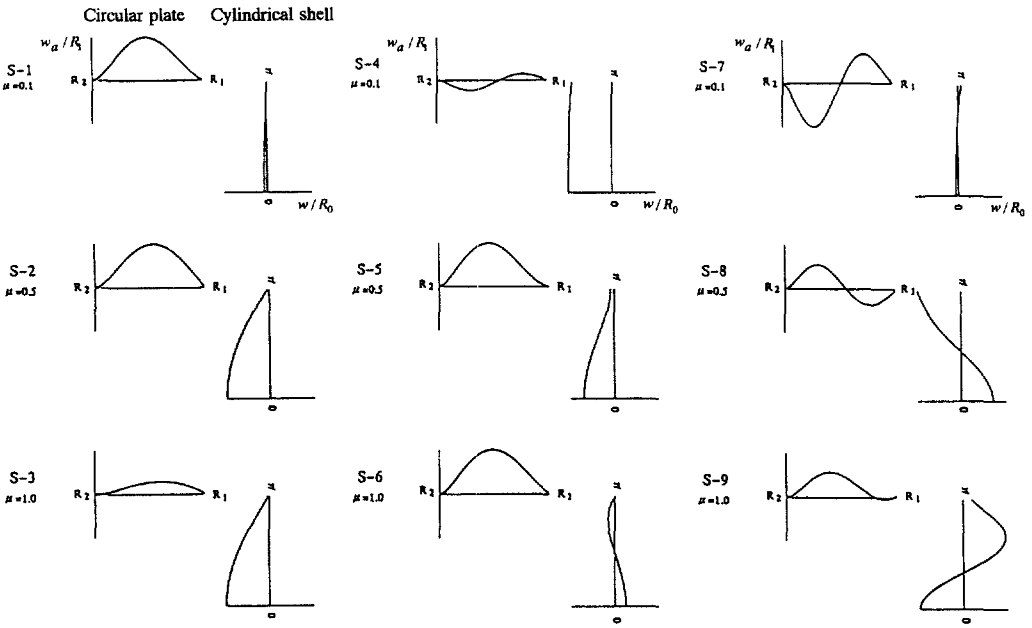

Fig. 18. Mode shapes of a vessel $N = 3, \beta = 20, n = 2, R_2/R_1 = 0.1$ (0°, 90°, 0°) (inner edges clamped: symmetric vibration).

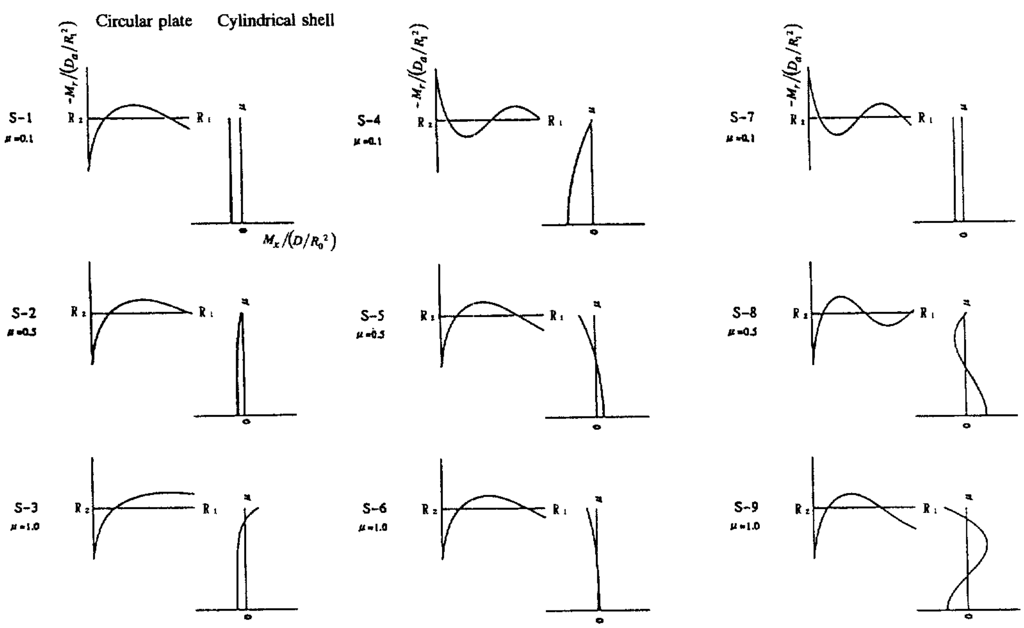

Fig. 19. Moment distribution of a vessel $N = 3, \beta = 20, n = 2, R_2/R_1 = 0.1$ (0°, 90°, 0°) (inner edges clamped: symmetric vibration).

are simple when the cylinder is long. The value of the bending moment is maximum at the inner clamped edge of the circular plate.

4. CONCLUDING REMARKS

In this paper, an analytical solution procedure has been developed for the free vibration of vessels consisting of a laminated composite circular cylinder and circular plate lids. As numerical examples, natural frequencies, mode shapes and bending moment distributions were found for vessels with symmetric cross-ply laminates, and from the results, its characteristics were clarified.

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APPENDIX A: SOLUTIONS OF EQUATIONS OF MOTION FOR A LAMINATED CROSS-PLY CIRCULAR PLATE

For out-of-plane vibration:

Solutions of eqn (24) can be expressed as follows:

$$w_a = w_n(r)\cos(n\theta) = \left(\sum_{m=0}^{\infty} X_m r^{\lambda+m}\right)\cos(n\theta) \tag{A1}$$

where X_m are undetermined coefficients. Substituting eqn (A1) into eqn (24), one obtains a fourth order characteristic equation for λ and the coefficients X_m $(m \geqq 1)$ are successively determined with X_0 left undetermined. When the roots λ are conjugate complex number $\xi \pm i\kappa$, the corresponding two independent solutions are expressed as follows:

$$w_n(r) = \sum_{m=0}^{\infty} [X_{m1}\cos(\kappa \ln r) \pm X_{m2}\sin(\kappa \ln r)]r^{\xi+m} \tag{A2}$$

For in-plane vibration:

Solutions of eqn (35) can be expressed as

$$u_a = u_n(r)\cos(n\theta) = \left(\sum_{m=0}^{\infty} Y_m r^{\lambda+m}\right)\cos(n\theta)$$

$$v_a = v_n(r)\sin(n\theta) = \left(\sum_{m=0}^{\infty} Z_m r^{\lambda+m}\right)\sin(n\theta) \tag{A3}$$

where Y_m and Z_m are undetermined coefficients. Substituting eqn (A3) into eqn (35), one obtains a fourth order characteristic equation for λ and the coefficients Y_m and Z_m are obtained in turn with Y_0 (or Z_0) left undetermined.

Table A1 shows the kind of roots λ for each characteristic equation.

Table A1. Roots λ of characteristic equation

Vibration	Circumferential wave number	Root
Out-of-plane	$n = 2$	4 Real
	$n \geqq 3$	2 Conjugate complex
In-plane	$n \geqq 2$	4 Real

APPENDIX B: COMPARISON WITH RESULTS BY OTHER METHOD

For laminated composite circular cylindrical shells, Hu *et al.* (1966) have obtained frequencies by a series solution. In the present theory, the frequency equation for symmetric vibrations of a laminated composite circular cylindrical shell with both ends free is obtained by

$$\partial L_{sn}/\partial a_{np} = 0 \quad (p = 1\text{–}4) \tag{B1}$$

and for antisymmetric vibrations, by

$$\partial L_{an}/\partial a^*_{np} = 0 \quad (p = 1\text{–}4) \tag{B2}$$

where L_{sn} and L_{an} are expressed by eqn (12). The frequency equation with both ends clamped is obtained by

$$\Delta_0 = 0 \quad \text{(symmetric vibration)} \tag{B3}$$

and

$$\Delta_0^* = 0 \quad \text{(antisymmetric vibration)} \tag{B4}$$

The well-known basic equations for out-of-plane vibrations of an isotropic circular plate are obtained by putting, in eqns (24)–(26),

$$D_{a11} = D_{a22} = 1, \quad D_{a12} = \nu, \quad D_{a66} = (1-\nu)/2, \quad E_{a0} = E/12(1-\nu^2), \quad 12\beta_a^2 = \beta \tag{B5}$$

where ν and E are the Poisson ratio and Young's modulus, respectively. The frequency equation with inner and outer edges free is obtained by

$$\partial L_{Bn}/\partial d_{np} = 0 \quad (p = 1\text{–}4) \tag{B6}$$

and with both edges clamped, by

$$\Delta_{B0} = 0 \tag{B7}$$

where L_{Bn} is expressed by eqn (33).

The basic equations for in-plane vibrations of an isotropic circular plate are obtained by putting, in eqns (35)–(37),

$$A_{a11} = A_{a22} = 12, \quad A_{a12} = 12\nu, \quad A_{a66} = 6(1-\nu), \quad E_{a0} = E/12(1-\nu^2), \quad 12\beta_a^2 = \beta \tag{B8}$$

The frequency equation with inner and outer edges free is obtained by

$$\partial L_{In}/\partial f_{np} = 0 \quad (p = 1\text{–}4) \tag{B9}$$

and with both edges clamped, by

$$\Delta_{I0} = 0 \tag{B10}$$

where L_{In} is expressed by eqn (44).

Table B1 presents a comparison of the values of $\sqrt{\alpha^4/\beta^2}$ by the present method with those by Hu *et al.* (1996) for a laminated composite circular cylindrical shell, and Tables B2 and B3 compare the values of α by the present methods with those by Bessel functions for an isotropic circular plate, respectively, provided that D_{a11} and D_{a22} are taken to be 0.9999 in eqn (B5) and A_{a11} and A_{a22} are 11.9999 in eqn (B8) to obtain four independent solutions. From the tables, one can see that the results from the present method are in close agreement with those obtained by the previous methods.

Table B1. Comparison of the values of $\sqrt{\alpha^4/\beta^2}$ by the present method with those by Hu *et al.* for a laminated composite circular cylindrical shell [($N = 3$, $n = 2$, $\mu = 1$, $\beta = 20$, (0°, 90°, 0°)]

		Both ends clamped		Both ends free	
	Mode	Present method	Hu *et al.*	Present method	Hu *et al.*
Symm vibration	1st	0.612	0.612	1.012	1.012
	2nd	1.971	1.971	2.147	2.147
Antisymm vibration	1st	1.246	1.246	1.694	1.694
	2nd	2.871	2.871	2.981	2.981

Table B2. Comparison of the values of α by the present method with those by Bessel functions for an isotropic plate (out-of-plane vibration $n = 2$, inner and outer edges clamped)

R_2/R_1	Mode	Bessel function	Present method
0.1	1st	6.0512	6.0512
	2nd	9.5104	9.5104
	3rd	12.928	12.928
0.2	1st	6.4668	6.4668
	2nd	10.321	10.321
	3rd	14.173	14.172

Table B3. Comparison of the values of α by the present method with those by Bessel functions for an isotropic circular plate without a hole [$n = 2$, β $(=12R_1^2/h^2) = 5000$ (in-plane vibration)]

	Boundary condition	Mode	Bessel function	Present method
Out-of-plane vibration	Clamped	1st	5.9057	5.9057
		2nd	9.1969	9.1969
	Free	1st	2.3148	2.3148
		2nd	5.9380	5.9380
In-plane vibration	Clamped	1st	14.679	14.679
		2nd	16.997	16.997
	Free	1st	9.9056	9.9055
		2nd	13.325	13.325